Journal of Global Optimization (2006) 35: 387–403
DOI 10.1007/s10898-005-3844-2

A Polytope Approach to the Optimal Assembly Problem

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(Received 17 February 2004; accepted in revised form 2 October 2005)

Abstract. The problem of assembling components into series modules to maximize the system reliability has been intensively studied in the literature. Invariably, the methods employed exploit special properties of the reliability function through standard analytical optimization techniques. We propose a geometric approach by exploiting the assembly polytope – a polytope generated by the potential assembly configurations. The new approach yields simpler proofs of known results, as well as new results about systems where the number of components in a module is not fixed, but subject to lower and upper bounds.

1. Introduction

Consider a monotone system with p modules where both the system and each module are in one of two states – *operative* or *inoperative*, and where the state of the system depends on the joint states of the modules. The adjective "monotone" means that the system's performance is monotone in the performance of its modules, that is, it cannot move from the operative state to the inoperative state as the result of one of the modules moving from the inoperative state to the operative state. Each module M_i requires components of different types that are in series structure, that is, M_i is operative if and only if all its components are operative. The system's state is determined by a *performance function* $J(.)$ whose variables are the states of the modules, that is, when the state of module M_i for each $i = 1, \dots, p$ is s_i, the state of the system is $J(s_1, \dots, s_p)$. By assigning value 1 to operativeness and value 0 to inoperativeness, the function J is a Boolean function, with domain $\{0, 1\}^p$ and range $\{0, 1\}$; system-monotonicity means that the function J is monotone. Constraints on the number of components of each type that the modules require are specified in terms of lower and upper bounds, and we refer to these constraints as the *system specifications*.

We next consider the components of which the modules are constructed. There are t types of components indexed $u = 1, \dots, t$, and for each u there are n_u components of type u. Let r_{uj} denote the *reliability* of the jth item of type u, $j=1,\dots,n_u$, $u=1,\dots,t$, that is, the probability that this item is operative. Operativeness of these items are assumed to be stochastically independent. Without loss of generality, we will assume that the components of each type are indexed so that

$$0<r_{u1}\leqslant r_{u2}\leqslant\cdots\leqslant r_{un_u}\leqslant 1 \qquad \text{for } u=1,\dots,t; \tag{1.1}$$

An *assembly* is an assignment of the n_u components of each type u to the modules that satisfies the system specifications; formally, such an assembly consists of partitions for each $u=1,\dots,t$ of the set $\{1,\dots,n_u\}$ into parts $\pi_{u1},\dots,\pi_{up}$. As a module is operative if and only if all its components are, the probability that module M_i is operative when π is applied is given by

$$r(\pi)_i=\prod_{u=1}^{t}\prod_{j\in\pi_{ui}} r_{uj}. \tag{1.2}$$

and we refer to $r(\pi)_i$ as the *reliability* of M_i *under* assembly π. The *system reliability under* assembly π, that is, the probability that the system is operative when π is applied, is then given by

$$R(\pi)\equiv\sum_{s\in\{0,1\}^p} J(s)\left\{\prod_{\{i:s_i=0\}}[1-r(\pi)_i]\right\}\left\{\prod_{\{i:s_i=1\}} r(\pi)_i\right\}. \tag{1.3}$$

The *optimal assembly problem* is to determine an assembly such that the system reliability is maximized over the set of feasible assemblies. A number of authors [2, 3, 5, 6, 7, 9, 11, 13, 14, 15, 16] studied instances of this problem where the system specifications prescribe the number of components of each type that the modules require (see [10] for a review of specific results obtained in these references), Hwang and Rothblum [10] solved this problem in its generality. Their solution method first considers systems with two modules, then extends the analysis to systems with arbitrarily many modules but with one item-type required by all the modules, and finally extends the analysis to the general case. Each step requires subtle perturbation arguments that involve part-reliabilities and the structure function.

In this paper, we treat the assembly problem as an assignment problem, namely, every component must be assigned to a module. We then seek an optimal assembly under lower and upper bound constraints on the number of parts of each type in each module. The need for assignment may come up in various scenarios, for example, when the components are personnel each must be assigned to a job, or when the storage of components outside of the modules is expensive, or when idle components may cause degrading in reliability (such as unused cars or uninhabited houses). Our solution method uses a new polytope-approach. The new approach yields simpler proofs of existing results for the problem with prescribed specifications as well as a solution for the problem where the number of components of each type that the modules require are constrained by lower and upper bounds, a problem which has not been solved before.

In Section 2, we formalize the reliability maximization problem as an *additive assembly problem with asymmetric Schur convex objective* – terms that are defined in that section. After introducing some preliminaries about supermodular functions in Section 3, we analyze and solve the additive assembly problem with asymmetric Schur convex objective with prescribed requirements and with bounds on the requirements in Sections 4 and 5, respectively. Finally, Sections 6 contains a discussion of potential extensions of the results of the earlier sections.

2. The Additive Assembly Problem with Asymmetric Schur Convex Objective

The ith *unit vector* in R^p is the vector e^i with $e^i_i = 1$ and $e^i_t = 0$ for $t \in \{1, \dots, p\} \backslash \{i\}$. With i ranging over $i = 1, \dots, p$, we refer to these vectors as the *standard unit vectors* in R^p. Also, for $i, j \in \{1, \dots, p\}$, let e^{ij} be the difference between the i- and the j-unit vectors in R^p.

Let T be an interval and g a real-valued continuously differentiable function on T^p, that is, g has continuous partial derivatives with respect to all variables. The function g is called *asymmetric Schur convex* if for every $y \in T^p$ and distinct $i, j \in \{1, \dots, p\}$, the function $g^{ij}_y(\gamma) \equiv g(y + \gamma \mathrm{e}^{ij})$ does not decrease after an increase (here "after" means – "as γ increases"); as g is continuously differentiable, this condition is equivalent to the requirement that $\left(\frac{\mathrm{d}}{\mathrm{d}\gamma}\right) g^{ij}_y = \left(\frac{\partial}{\partial x_i}\right) g(y + \gamma \mathrm{e}^{ij}) - \left(\frac{\partial}{\partial x_j}\right) g(y + \gamma \mathrm{e}^{ij})$ does not have a sign-switch from a positive value to a negative value as γ increases. The function g is called *strictly asymmetric Schur convex* if it is asymmetric Schur convex and none of the functions g^{ij}_y is constant on an interval; as g is continuously differentiable, the latter means that the derivatives of these functions cannot vanish on any interval. Asymmetric Schur convex functions were introduced in [12]; they generalize (quasi) convex and (regular) Schur convex functions and have the following important property (see [12]):

PROPOSITION 2.1. *Let T be an interval and let g be an asymmetric Schur convex function on T^p. If P is a polytope contained in T^p where the directions of the edges of P are differences of pairs of standard unit vectors in R^p, then g attains a maximum at one of P's vertices. Further, if g is strictly asymmetric Schur convex, then every maximizer of g over P is a vertex of P.*

We next show that the assembly problem described in the introduction fits within a framework of optimization problems that concern asymmetric Schur convex functions. For each $u=1,\dots,t$ and $j=1,\dots,n_u$, define $\rho_{uj}=\ln r_{uj}$. As the indexing of r_{uj}'s satisfies (1.1), we have

$$-\infty<\rho_{u1}\leqslant\rho_{u2}\leqslant\cdots\leqslant\rho_{un_u}\leqslant 0 \qquad \text{for } u=1,\dots,t. \tag{2.1}$$

For each assembly π and $i=1,\dots,p$, let

$$\rho(\pi)_i\equiv\ln r(\pi)_i=\ln\left[\prod_{u=1}^{t}\prod_{j\in\pi_{ui}} r_{uj}\right]=\sum_{u=1}^{t}\sum_{j\in\pi_{ui}}\rho_{uj}. \tag{2.2}$$

Thus, each assembly π is associated with the two p-dimensional vectors $r(\pi)=[r(\pi)_1,\dots,r(\pi)_p]$ and $\rho(\pi)=[\rho(\pi)_1,\dots,\rho(\pi)_p)]$.

Given the performance function $J:\{0,1\}^p\to\{0,1\}$, let f and g be the real-valued functions on $(0,1]^P$ and $(-\infty,0]^p$, respectively, with

$$f(x)=\sum_{s\in\{0,1\}^p}J(s)\left[\prod_{\{i:s_i=0\}}(1-x_i)\right]\left[\prod_{\{i:s_i=1\}}x_i\right] \quad \text{for every } x\in(0,1]^p, \tag{2.3}$$

and

$$\begin{aligned} g(y)&=f[\exp(y_1),\dots,\exp(y_p)]\\ &=\sum_{s\in\{0,1\}^p}J(s)\left[\prod_{\{i:s_i=0\}}(1-\mathrm{e}^{y_i})\right]\left[\prod_{\{i:s_i=1\}}\mathrm{e}^{y_i}\right] \quad \text{for every } y\in(-\infty,0]^p. \end{aligned} \tag{2.4}$$

Using (1.3) and (2.2), the system reliability under an assembly π is expressible by

$$R(\pi)=\sum_{s\in\{0,1\}^p}J(s)\left\{\prod_{i:s_i=0}[1-r(\pi)_i]\right\}\left\{\prod_{i:s_i=1}r(\pi)_i\right\}=f[r(\pi)]=g[\rho(\pi)]. \tag{2.5}$$

Hwang and Rothblum [10, Example 3] estabished the following result.

PROPOSITION 2.2. *Given any monotone performance function* $J:\{0,1\}^p\to\{0,1\}$, *the function g defined by* (2.3)–(2.4) *is asymmetric Schur convex on* $(-\infty,0]^p$.

Proposition 2.2 implies that the problem of determining an assembly which maximizes system-reliability over a set of assemblies $\prod$ is an instance of the problem of maximizing an expression $g[\rho(\pi)]$ over $\prod$ with $\rho(\pi)_i = \sum_{u=1}^{t} \sum_{j \in \pi_{ui}} \rho_{uj}$ for $i = 1, \ldots, p$, with the ρ_{uj}'s satisfying (2.1), and with $g(.)$ as an asymmetric Schur convex on $(-\infty, 0]^p$. Henceforth, we consider this more general problem to which we refer as the *additive assembly problem with asymmetric Schur convex objective* (in this context, the term *multi-partition* is used in the literature synonymously with assembly). We will solve this problem when feasible assemblies π are those for which the size of the π_{ui}'s must satisfy lower and upper bounds.

3. Preliminaries on Supermodular Functions

Given a real-valued function λ on the subsets of $\{1, \ldots, p\}$ with $\lambda(\emptyset) = 0$, we define in this section two polytopes in R^p. A *permutation of* $\{1, \ldots, p\}$ is a vector $\sigma = (i_1, \ldots, i_p)$ with $\{i_1, \ldots, i_p\}$ distinct and with $\{i_1, \ldots, i_p\} = \{1, \ldots, p\}$. Each permutation σ defines a p-vector λ_σ with

$$(\lambda_\sigma)_{i_s} = \lambda(\{i_1, \ldots, i_s\}) - \lambda(\{i_1, \ldots, i_{s-1}\}) \quad \text{for } s = 1, \ldots, p. \tag{3.1}$$

The *permutation polytope corresponding to* λ, denoted H^λ, is the convex hull of the λ_σ's, with σ ranging over the set $\sum^p$ of all permutations of $\{1, \ldots, p\}$. Also, the polytope C^λ is defined as the solution set of the linear inequality system

$$\sum_{i \in \mathrm{I}} x_i \geqslant \lambda(\mathrm{I}) \quad \text{for each nonempty subset } I \text{ of } \{1, \ldots, p\}, \tag{3.2}$$

and

$$\sum_{i=1}^{p} x_i = \lambda(\{1, \ldots, p\}). \tag{3.3}$$

A real-valued function λ on the nonempty subsets of $\{1, \ldots, p\}$ with $\lambda(\emptyset) = 0$ is called *supermodular* if for every pair I and J of subsets of $\{1, \ldots, p\}$,

$$\lambda(I \cup J) + \lambda(I \cap J) \geqslant \lambda(I) + \lambda(J); \tag{3.4}$$

λ is called *strictly supermodular* if strict inequality holds whenever the two sets are not ordered by set inclusion, that is, $I \not\subseteq J$ and $J \not\subseteq I$.

Parts (i) and (ii) of the next result are due to Shapley [18], see also, [8]; for part (iii) see [9].

PROPOSITION 3.1. *Suppose* λ *is supermodular on the subsets of* $\{1, \dots, p\}$. *Then:*

(i) $H^{\lambda} = C^{\lambda}$,
(ii) *the vertices of* $H^{\lambda} = C^{\lambda}$ *are precisely the* λ_{σ}*'s where* σ *ranges over* $\sum^{p}$, *and*
(iii) *each direction of an edge of* $H^{\lambda} = C^{\lambda}$ *is proportional to the difference of a pair of standard unit vectors in* R^{p}.

4. Prescribed-Sizes Problem

In this section we assume that nonnegative integers $\{n_{ui} : u = 1, \dots, t$ and $i = 1, \dots, p\}$ are given, and the system specifications require that the number of components of (each) type u in (each) module M_i must equal n_{ui}.

An assembly is called *monotone* if there exists a module M_{i_1} taking the n_{ui_1} most reliable items of each type u , a second module M_{i_2} taking the n_{ui_2} next most reliable items of each type u, and so on, until the last module M_{i_p} takes the n_{ui_p} least reliable items of each type u.

Let $\prod$ be the set of all assemblies π with each π_{ui} having the prescribed size n_{ui}. The *assembly polytope*, denoted P, is defined as the convex hull of the $\rho(\pi)$'s, with π ranging over $\prod$. Also, for each subset I of $\{1, \dots, p\}$ let

$$\rho_*(I) = \min_{\pi \in \prod} \sum_{i \in I} \rho(\pi)_i; \tag{4.1}$$

in particular, $\rho_*(\emptyset) = 0$ and

$$\rho_*(\{1, \dots, p\}) = \sum_{u=1}^{t} \sum_{j=1}^{n_u} \rho_{uj}. \tag{4.2}$$

For a subset I of $\{1, \dots, p\}$ and $u = 1, \dots, t$, let $n_u(I) = \sum_{i \in I} n_{ui}$; we then have that

$$\rho_*(I) = \sum_{u=1}^{t} \sum_{j=1}^{n_u(I)} \rho_{uj}. \tag{4.3}$$

We observed that the $n_u(I)$s (for all subserts I of $\{1, \dots, p\}$) and the sums $\sum_{j=1}^{s} \rho_{uj}$s (for $s = \{1, \dots, n\}$) can be determined, respectively, by at most 2^p and $\sum_{u=1}^{p} n_u = n$ additions. Given these quantities, the $\sum_{j=1}^{n_u(I)} \rho_{ui}$s are available for each u and for each I, and (4.3) can be used to determine the $\rho_*(I)$'s with $t-1$ extra additions for each I. The total computational effort for computing all the $\rho_*(I)$'s in this way amounts to $t2^p + n$ additions. With p fixed, the bound is linear in the number of components n.

LEMMA 4.1. *ρ_* is supermodular.*

Proof. For subsets I and J of $\{1,\ldots,p\}$ and $u=1,\ldots,t, n_u(I\cup J)-n_u(I)=n_u(J\setminus I)=n_u(J)-n_u(I\cap J)$, and (2.1) implies that

$$\begin{aligned}\rho_*(I\cup J)-\rho_*(I) &= \sum_{u=1}^{t}\left[\sum_{i=1}^{n_u(I\cup J)}\rho_{ui}-\sum_{i=1}^{n_u(I)}\rho_{ui}\right]=\sum_{u=1}^{t}\left[\sum_{i=n_u(I)+1}^{n_u(I\cup J)}\rho_{ui}\right]\\ &=\sum_{u=1}^{t}\left[\sum_{i=n_u(I)+1}^{n_u(I)+n_u(J\setminus I)}\rho_{ui}\right]\geqslant\sum_{u=1}^{t}\left[\sum_{i=n_u(I\cap J)+1}^{n_u(I\cap J)+n_u(J\setminus I)}\rho_{ui}\right]\\ &=\sum_{u=1}^{t}\left[\sum_{i=n_u(I\cap J)+1}^{n_u(J)}\rho_{ui}\right]=\sum_{i=1}^{t}\left[\sum_{i=1}^{n_u(J)}\rho_{ui}-\sum_{i=1}^{n_u(I\cap J)}\rho_{ui}\right]\\ &=\rho_*(J)-\rho_*(I\cap J).\end{aligned}$$

□

Given a permutation $\sigma=(i_1,\ldots,i_p)$ of $\{1,\ldots,p\}$, the assembly corresponding to σ, denoted π_σ, is defined as the assembly with

$$(\pi_\sigma)_{ui_s}=\{n_u(\{i_1,\ldots,i_{s-1}\})+1,\ldots,n_u(\{i_1,\ldots,i_s\})\}\quad \text{for } u=1,\ldots,t \text{ and } s=1,\ldots,p. \tag{4.4}$$

LEMMA 4.2

(i) *For every permutation σ of $\{1,\ldots,p\}, \pi_\sigma$ is a monotone assembly with $\rho(\pi_\sigma)=(\rho_*)_\sigma$.*
(ii) *The correspondence $\sigma\to\pi_\sigma$ mapping $\sum^p$ into monotone assemblies is onto.*

Proof. (i) Consider a permutation $\sigma=(i_1,\ldots,i_p)$ of $\{1,\ldots,p\}$. The monotonicity of π_σ is immediate. Next, for $u=1,\ldots,t$ and $s=1,\ldots,p$, let $n_u(s)\equiv n_u(\{i_1,\ldots,i_s\})=\sum_{q=1}^{s}n_{ui_q}$. It follows from (4.3) and (4.4) that for $s=1,\ldots,p$, $\rho_*(\{i_1,\ldots,i_s\})=\sum_{u=1}^{t}\sum_{j=1}^{n_u(s)}\rho_{uj}=\sum_{q=1}^{s}[\rho(\pi_\sigma)]_{i_q}$, and therefore $[(\rho_*)_\sigma]_{i_s}=\rho_*(\{i_1,\ldots,i_s\})-\rho_*(\{i_1,\ldots,i_{s-1}\})=[\rho(\pi_\sigma)]_{i_s}$. As $\{i_1,\ldots,i_p\}=\{1,\ldots,p\}$, we conclude that $(\rho_*)_\sigma=\rho(\pi_\sigma)$.

(ii) A monotone assembly π, defines a ranking on $\{1,\ldots,p\}$ which defines a permutation $\sigma=(i_1,\ldots,i_p)$ with i_s as the sth element in the ranking. It is straightforward to see that in this case $\pi_\sigma=\pi$. □

As ρ_* is a function on the subsets of $\{1,\ldots,p\}$ with $\rho_*(\emptyset)=0$, the polytopes C^{ρ_*} and H^{ρ_*} are well-defined (see Section 2).

THEOREM 4.3

(i) $P = H^{\rho_*} = C^{\rho_*}$.
(ii) *The vertices of P are the vectors* $\{\rho_\pi : \pi$ *ranging over the set of monotone assemblies*$\}$.
(iii) *Each direction of an edge of P is proportional to the difference of a pair of standard unit vectors in* R^p.

Proof. From (4.1), for every assembly π and subset I of $\{1, \dots , p\}$, $\rho_*(I) \leqslant \sum_{i\in I} \rho(\pi)_i$ and, using (4.2), $\rho_*(\{1, \dots , p\}) = \sum_{u=1}^{t} \sum_{j=1}^{n_u} \rho_{u_j} = \sum_{u=1}^{t} \sum_{i=1}^{p} \sum_{j\in\pi_i} \rho_{u_j} = \sum_{i=1}^{p} \rho(\pi)_i$. So, $\rho(\pi) \in C^{\rho_*}$. It follows that the convex hulls of the $\rho(\pi)$'s, namely P, is contained in C^{ρ_*}. Next, Lemma 4.2 implies that

$$P = \text{conv } \{\rho(\pi); \pi \text{ is an assembly }\} \supseteq \text{ conv } \{\rho(\pi); \pi \text{ is a monotone assembly }\} = \text{conv } \{(\rho_*)_\sigma; \sigma \text{ is a permutation of } \{1, \dots , p\}\} = H^{\rho_*}.$$

Next, Proposition 3.1 and the supermodularity of ρ_* (Lemma 4.1) imply that $H^{\rho_*} = C^{\rho_*}$ and that the vertices of this polytope are the $(\rho_*)_\sigma$s with σ ranging over the permutations of $\{1, \dots , p\}$. So, $P \subseteq C^{\rho_*} = H^{\rho_*} \subseteq P$, implying that $P = C^{\rho_*} = H^{\rho_*}$; further, Lemma 4.2 implies that the vertices of this polytope are the $\rho(\pi)$s with π ranging over the monotone assemblies. Finally, part (iii) follows directly from the third part of Proposition 3.1 and the above results. □

The linear inequality representation of P through C^{ρ_*} has p variables and 2^p constraints that use the $\rho_*(I)$'s. We recall the paragraph following (4.3) that shows how to determine the $\rho_*(I)$'s with $t2^p + n$ additions.

THEOREM 4.4. *There exists a monotone optimal assembly.*

Proof. Proposition 2.1, the asymmetric Schur convexity of g and part (iii) of Theorem 4.3 assure that g attains a maximum over P at a vertex of that polytope; Theorem 4.3 also assures that such a vertex has the representation $\rho(\pi^*)$ for some monotone assembly π^*. Now, let π be an arbitrary assembly; then $\rho(\pi) \in P$ and $R(\pi) = g[\rho(\pi)] \leqslant g[\rho(\pi^*)] = R(\pi^*)$. □

Theorem 4.4 implies that the assembly problem with prescribed sizes can be solved by evaluating the monotone assemblies; by Lemma 4.2, these are precisely the $p!\,\pi_\sigma$'s, determined by the permutations over $\{1, \dots , p\}$. In order to evaluate $R(\pi_\sigma) = g[\rho(\pi_\sigma)]$ for a permutation σ, one has to evaluate $g(.)$ at $\rho(\pi_\sigma) = (\rho_*)_\sigma$ (the equality following from Lemma 4.2). Now, each $(\rho_*)_\sigma$ is available by executing t subtractions of corresponding $\rho_*(I)$s;

again, see the paragraph following (4.3) for a discussion of the computation of all $\rho_*(I)$s using at most $t2^p+n$ additons.

5. The Bounded-Sizes Problem

In the current section we consider the optimal assembly problem where lower and upper bounds are prescribed on the number of components of each type to be assigned to each module. Specifically, throughout this section, integers L_{ui} and U_{ui} for $u=1,\dots,t$ and $i=1,\dots,p$ are given with $L_{ui}\leqslant U_{ui}$ and $\sum_{i=1}^{p} L_{ui}\leqslant n_u\leqslant\sum_{i=1}^{p} U_{ui}$ for $u=1,\dots,t$. We let $\prod^{L,U}$ be the set of assemblies π with $L_{ui}\leqslant|\pi_{ui}|\leqslant U_{ui}$ for $u=1,\dots,t$ and $i=1,\dots,p$, and the assembly polytope $P^{L,U}$ be the convex hull of the vectors $\rho(\pi)$ with π ranging over $\prod^{L,U}$.

In this section we show that an asymmetric Schur convex function $g(.)$ over $P^{L,U}$ attains a maximum at a vertex of $P^{L,U}$ and that vertices correspond to monotone assemblies; further, we provide a linear inequality representation of the polytope $P^{L,U}$.

The next result asserts optimality of monotone assemblies and assures a representation of vertices of $P^{L,U}$ via monotone assemblies. It follows from the results of Section 4.

THEOREM 5.1

(i) *There exists a monotone assembly which is optimal over* $\prod^{L,U}$.
(ii) *Each vertex of* $P^{L,U}$ *has a representation* $\rho(\pi)$ *with* π *as a monotone assembly in* $\prod^{L,U}$.

Proof. As there are finitely many assemblies, there exists an optimal assembly. Let π^* be an optimal assembly. By considering the problem with prescribed number $|(\pi^*)_{ui}|$ of components of each type u for each module M_i, we have from Theorem 4.4 that this problem has a monotone optimal assembly, say π'. It follows $\rho(\pi^*)\leqslant\rho(\pi')$; the optimality of π^* for the bounded-sizes problem then implies that $\rho(\pi')=\rho(\pi^*)$, assuring that π' is also optimal for that problem.

A vertex v of $P^{L,U}$ has a representation $\rho(\bar\pi)$ with $\bar\pi\in\prod^{L,U}$. Let P be the polytope corresponding to the assembly problem where the number of components of each type u for each module M_i is $|\bar\pi_{ui}|$. As P is the convex hull of a set that is smaller than the one defining $P^{L,U}$ and as $v\in P$, v is a vertex of P as well. By Theorem 4.3, v has a representation as $\rho(\pi')$ where π' is a monotone assembly having $|\pi'_{ui}|=|\bar\pi_{ui}|$ for each u and i, in particular, $\pi'\in\prod^{L,U}$. □

The conclusions and proof of Theorem 5.1 extend to sets of assemblies with arbitrary constraints on the number of components of each type in each module (not just constraints that are determined through lower and upper bounds). In the remainder of this section we explore the extra structure available in bounded-sizes problems to obtain a refined analysis.

Parallel to our development in Section 4, define the real-valued function on subsets of $\{1, \dots, p\}$ where for subset I,

$$\rho_*^{L,U}(I) = \min\left\{\sum_{i\in I} \rho(\pi)_i : \pi \in \prod^{L,U}\right\}; \tag{5.1}$$

in particular, $\rho_*^{L,U}(\emptyset) = 0$ and

$$\rho_*^{L,U}(\{1, \dots, p\}) = \sum_{u=1}^{t}\sum_{j=1}^{n_u} \rho_{u_j}. \tag{5.2}$$

For each subset I of $\{1, \dots, p\}$, let

$$n_u^-(I) = \min\left\{\sum_{i\in 1} U_{ui}, n_u - \sum_{i\in I^c} L_{ui}\right\}, \quad \text{for } u = 1, \dots, t. \tag{5.3}$$

where $I^c \equiv \{1, \dots, p\}\backslash I$. The next lemma provides a representation of $\rho_*^{L,U}(.)$ that resembles (4.3) (which applied to the fixed sizes case).

LEMMA 5.2

(i) *For each permutation* $\sigma = (i_1, \dots, i_p)$ *of* $\{1, \dots, p\}$, *there exists a monotone assembly* $\pi_\sigma \in \prod^{L,U}$ *with* $(\rho_*^{L,U})_\sigma = \rho(\pi)$ *and*

$$(\pi_\sigma)_{ui_s} = \{n_u^-(i_1, \dots, i_{s-i}\}) + 1, \dots, n_u^-(\{i_1, \dots, i_s\})\}$$
$$\textit{for } u = 1, \dots, t \textit{ and } s = 1, \dots, p. \tag{5.4}$$

(ii) *For each subset* I *of* $\{1, \dots, p\}$,

$$\rho_*^{L,U}(I) = \sum_{u=1}^{t}\sum_{j=1}^{n_u^-(I)} \rho_{uj}. \tag{5.5}$$

Proof. (i) For each assembly $\pi \in \prod^{L,U}$ and $u = 1, \dots, t$, $\sum_{i\in I} |\pi_{ui}| \leqslant \sum_{i\in I} U_{ui}$ and $\sum_{i\in I} |\pi_{ui}| = \sum_{i=1}^{p} |\pi_{ui}| - \sum_{i\in I^c} |\pi_{ui}| \leqslant n - \sum_{i\in I^c} L_{ui}$, implying that $\sum_{i\in I} |\pi_{ui}| \leqslant n_u^-(I)$; hence, (2.1) (which includes the assertion that the ρ_{uj}'s are nonpositive) implies that $\sum_{i\in I}\sum_{j\in\pi_{ui}} \rho_{uj} \geqslant \sum_{j=1}^{n_u^-(I)} \rho_{uj}$. It follows that

$$\sum_{i\in I}[\rho(\pi)]_i = \sum_{i\in I}\sum_{u=1}^{t}\sum_{j\in\pi_{u_i}} \rho_{uj} = \sum_{u=1}^{t}\sum_{i\in I}\sum_{j\in\pi_{u_i}} \rho_{uj} \geqslant \sum_{u=1}^{t}\sum_{j=1}^{n_u^-(I)} \rho_{uj},$$

and therefore

$$\rho_*^{L,U}(I) \geqslant \sum_{u=1}^{t} \sum_{j=1}^{n_u^-(I)} \rho_{uj}. \tag{5.6}$$

Let $\sigma = (i_1, \ldots, i_p)$ be a permutation of $\{1, \ldots, p\}$. For $u = 1, \ldots, t$ and $i = 1, \ldots, p$ define n_{ui}^- by setting for each $s = 1, \ldots, p, n_{ui_s}^- \equiv n_u^-(\{i_1, \ldots, i_s\}) - n_u^-(\{i_1, \ldots, i_{s-1}\})$. Fix $u \in \{1, \ldots, p\}$. We will show that for $u = 1, \ldots, t$ and $i = 1, \ldots, p, L_{ui} \leqslant n_{ui}^- \leqslant U_{ui}$; this will be established by showing that for $u = 1, \ldots, t$ and $s = 1, \ldots, p, L_{ui_s} \leqslant n_{ui_s}^- \leqslant U_{ui_s}$. So, fix u and s and set $I_{s-1} \equiv \{i_1, \ldots, i_{s-1}\}$ and $I_s = \{i_1, \ldots, i_s\}$. We consider two cases, one of which has two subcases.

Case I: $\mathbf{n_u^-(I_{s-1}) = \sum_{i \in I_{s-1}} U_{ui} \leqslant n - \sum_{i \in I_{s-1}^c} L_{ui}}$. In this case, if $n_u^-(I_s) = \sum_{i \in I_s} U_{ui}$, then $n_{ui_s}^- = n_u^-(I_s) - n_u^-(I_{s-1}) = U_{ui_s} \geqslant L_{ui_s}$, assuring that $L_{ui_s} \leqslant n_{ui_s}^- \leqslant U_{ui_s}$. Alternatively, if $n_u^-(I_s) = n - \sum_{i \in I_s^c} L_{ui} \leqslant \sum_{i \in I_s} U_{ui}$, then $n_{ui_s}^- = n_u^-(I_s) - n_u^-(I_{s-i}) \leqslant \sum_{i \in I_s} U_{ui} - \sum_{i \in I_{s-1}} U_{ui} = U_{ui_s}$ and $n_{ui_s}^- = n_u^-(I_s) - n_u^-(I_{s-1}) \geqslant (n - \sum_{i \in I_s^c} L_{ui}) - (n - \sum_{i \in I_{s-1}^c} L_{ui}) = L_{ui_s}$.

Case II: $\mathbf{n_u^-(I_{s-1}) = n - \sum_{i \in I_{s-1}^c} L_{ui} \leqslant \sum_{i \in I_{s-1}} U_{ui}}$. In this case, $\sum_{i \in I_s} U_{ui} = \sum_{i \in I_{s-1}} U_{ui} + U_{ui_s} \geqslant (n - \sum_{i \in I_{s-1}}^{c} L_{ui}) + L_{ui_s} = n - \sum_{i \in I_s^c} L_{ui}$, assuring that $n_u^-(I_s) = n - \sum_{i \in I_s^c} L_{ui}$ and $n_{ui_S}^- = n_u^-(I_s) - n_u^-(I_{s-1}) = (n - \sum_{i \in I_s^c} L_{ui}) - (n - \sum_{i \in I_{s-1}^c} L_{ui}) = L_{ui_s} \leqslant U_{ui_s}$, assuring that $L_{ui_s} \leqslant n_{ui_s}^- \leqslant U_{ui_s}$.

Consider the assembly problem with prescribed sizes n_{ui}^- for $u = 1, \ldots, t$ and $i = 1, \ldots, p$ and the monotone assembly π_σ corresponding to σ as defined by (4.4), that is, π_σ is determined by (5.7). In particular, $|(\pi_\sigma)_{ui}| = n_{ui}^-$ for each u and i, and the above paragraphs assure that $\pi_\sigma \in \prod^{L,U}$. Next, for $s = 1, \ldots, p, \cup_{i \in I_s} (\pi_\sigma)_{ui} = \{1, \ldots, n_u^-(I_s)\}$ and

$$\begin{aligned} \sum_{i \in I_s} [\rho(\pi_\sigma)]_i &= \sum_{i \in I_s} \sum_{u=1}^{t} \sum_{j \in (\pi_\sigma)_{ui}} \rho_{uj} = \sum_{u=1}^{t} \sum_{i \in I_s} \sum_{j \in (\pi_\sigma)_{ui}} \rho_{uj} \\ &= \sum_{u=1}^{t} \sum_{j=1}^{n_u^-(I_s)} \rho_{uj} \leqslant \rho_*^{L,U}(I_s), \end{aligned} \tag{5.7}$$

the last inequality following form (5.6). As the definition of $\rho_*^{L,U}$ in (5.1) assures that $\rho_*^{L,U}(I_s) \leqslant \sum_{i \in I_s} [\rho(\pi_\sigma)]_i$, we conclude that equality holds throughout (5.7). Thus, for $s = 1, \ldots, p, \sum_{i \in I_s} [\rho(\pi_\sigma)]_i = \rho_*^{L,U}(I_s) = \sum_{i \in I_s} [(\rho_*^{L,U})_\sigma]_i$. It follows that for $s = 1, \ldots, p, \rho(\pi_\sigma) i_s = [(\rho_*^{L,U})_\sigma]_{i_s}$; as $\{i_1, \ldots, i_p\} = \{1, \ldots, p\}$, we conclude that $\rho(\pi_\sigma) = (\rho_*^{L,U})_\sigma$.

(ii) Let $I \subseteq \{1, \ldots, p\}$, say $I = \{i_1, \ldots, i_s\}$. One can construct a permutation σ of $\{1, \ldots, p\}$ with $i_1, \ldots, i_s$ as the first s elements, in order. With respect to the notation of the proof of (i), we have that $I = I_S$ and the established equalities in (5.7) prove (5.5). □

Part (i) of Lemma 5.2 generalizes part (i) of Lemma 4.2 from the prescribed-sizes case to the bounded-sizes case. But, part (ii) of Lemma 4.2 does not have a corresponding generalization as the number of monotone assemblies is generally larger than the number of permutations of $\{1, \dots, p\}$ which equals $p!$. In fact, for each (feasible) collection $\{n_{ui} : L_{ui} \leqslant n_{ui} \leqslant U_{ui}, u = 1, \dots, t \text{ and } i = 1, \dots, p\}$ there exists a monotone assembly for each permutation σ of $\{1, \dots, p\}$; with $\Gamma^{L,U}$, as the set of such feasible collections, the number of monotone assemblies is $|\Gamma| p!$. When $|\Gamma| \geqslant 1$, this number is larger than the number $p!$. It is noted that the effort of determining the vector associated with each monotone assembly is tn, so the total effort in computing all vectors associated with monotone assemblies is $tn|\Gamma|p!$.

COROLLARY 5.3. *Suppose* $I_1, \dots, I_k$ *are subsets of* $\{1, \dots, p\}$ *with* $\emptyset \subseteq I_1 \subset \cdots \subset I_k \subseteq \{1, \dots, p\}$. *Then there exists an assembly* $\pi \in \prod^{L,U}$ *satisfying* $\rho_*^{L,U}(I_t) = \sum_{i \in I_t} [\rho(\pi)]_i$.

Proof. We first assume that $I_1 \neq \emptyset$ and $I_k \neq \{1, \dots, p\}$. The list $I_1, \dots, I_k$ can be extended by adding sets to one having $p-1$ elements, that is, we can assume that $k = p-1$. Let $I_0 = \emptyset$ and $I_p = \{1, \dots, p\}$. We observe that in this case $\sigma \equiv (I_1 \setminus I_0, \dots, I_p \setminus I_{p-1})$ is a permutation. (here we identify set containing a single element with that element). Let $\pi \in \prod^{L,U}$ be a monotone assembly satisfying $\rho(\pi) = (\rho_*^{L,U})_\sigma$ (whose existence was established in Lemma 5.1). Now, suppose $\sigma = (i_1, \dots, i_p)$. For each $t = 1, \dots, p$ then $I_t = \{i_1, \dots, i_t\}$ and

$$\sum_{i \in I_t} [\rho(\pi)]_i = \sum_{s=1}^{t} [\rho(\pi)]_{i_s} = \sum_{s=1}^{t} [(\rho_*^{L,U})_\sigma]_{i_s} = \rho_*^{L,U}(\{i_1, \dots, i_t\}) = \rho_*^{L,U}(I_t).$$

The extension to the cases where $I_1 \neq \emptyset$ and/or $I_k \neq \{1, \dots, p\}$ is straightforward. □

LEMMA 5.4. $\rho_*^{L,U}$ *is supermodular.*

Proof. Let I and J be subsets of $\{1, \dots, p\}$ which are not ordered by set inclusion. Then $\emptyset \subseteq I \cap J \subset I \cup J \subseteq \{1, \dots, p\}$ and Corollary 5.3 implies that there exists an assembly π in $\prod^{L,U}$ satisfying $\rho_*^{L,U}(I \cap J) = \sum_{i \in I \cap J} [\rho(\pi)]_i$ and $\rho_*^{L,U}(I \cup J) = \sum_{i \in I \cup J} [\rho(\pi)]_i$. Consider the assembly problem with prescribed sizes $n_{ui} \equiv |\pi_{ui}|$ for $u = 1, \dots, t$ and $i = 1, \dots, p$ and let ρ_* be given by (4.1). For each $S \subseteq \{1, \dots, p\}$, $\rho_*^{L,U}(S) \leqslant \rho_*(S) \leqslant \sum_{i \in S} [\rho(\pi)]_i$. Thus, the assumption about π implies that $\rho_*^{L,U}(I \cap J) = \rho_*(I \cap J) = \sum_{i \in I \cap J} [\rho(\pi)]_i$

and $\rho_*^{L,U}(I \cup J) = \rho_*(I \cup J) = \sum_{i \in I \cup J}[\rho(\pi)]_i$. From the supermodularity of ρ_* (Lemma 4.1), then

$$\rho_*^{L,U}(I \cup J) + \rho_*^{L,U}(I \cap J) = \rho_*(I \cup J) + \rho_*(I \cap J) \geqslant \rho_*(I) + \rho_*(J) \geqslant \rho_*^{L,U}(I) + \rho_*^{L,U}(J),$$

proving that $\rho_*^{L,U}$ is supermodular. □

Recall that the set of permutations of $\{1, \ldots, p\}$ is denoted $\sum^p$. Given σ in $\sum^p$, the monotone assemblies in $\prod^{L,U}$ that is constructed in Lemma 5.2 (determined by (5.4)) will be denoted π_σ.

THEOREM 5.5

(i) $P^{L,U} = H^{\rho_*^{L,U}} = C^{\rho_*^{L,U}}$.
(ii) *The vertices of* $P^{L,U}$ *are precisely the* $\rho(\pi_\sigma)$*s where* σ *ranges over* $\sum^p$.
(iii) *Each direction of an edge of* $P^{L,U}$ *is proportional to the difference of a pair of standard unit vectors in* R^P.

Proof. (i) From (5.1), for every assembly $\pi \in \prod^{L,U}$ and subset I of $\{1, \ldots, p\}, \sum_{i \in I} \rho(\pi)_i \geqslant \rho_*^{L,U}(I)$, and from (5.2), $\sum_{i=1}^p \rho(\pi)_i = \sum_{u=1}^t \sum_{i=1}^p \sum_{j \in \pi_i} \rho_{uj} = \sum_{u=1}^t \sum_{j=1}^{n_u} \rho_{uj} = C^{\rho_*^{L,U}}(\{1, \ldots, p\})$, proving that $\rho(\pi) \in C^{\rho_*^{L,U}}$. It follows that the convex hulls of the $\rho(\pi)$s, namely $P^{L,U}$, is contained in $C^{\rho_*^{L,U}}$, that is, $P^{L,U} \subseteq C^{\rho_*^{L,U}}$. Next, the supermodularity of $\rho_*^{L,U}$ (Lemma 5.4) and Proposition 3.1 imply that $H^{\rho_*^{L,U}} = C^{\rho_*^{L,U}}$. Finally, Lemma 5.2 implies that

$$\begin{aligned} P^{L,U} &= \text{conv } \{\rho(\pi); \pi \in \Pi^{L,U}\} \supseteq \text{conv}\{\rho(\pi); \pi \\ &\quad \text{is a monotone assembly in } \Pi^{L,U}\} \\ &\supseteq \text{conv}\left\{(\rho_*^{L,U})_\sigma; \sigma \in \sum_p\right\} = H^{\rho_*^{L,U}}, \end{aligned}$$

So, $P^{L,U} \subseteq C^{\rho_*^{L,U}} = H^{\rho_*^{U,L}} \subseteq P^{L,U}$, implying that $P^{L,U} = C^{\rho_*^{L,U}} = H^{\rho_*^{U,L}}$.

(ii) The supermodularity of $\rho_*^{L,U}$ (Lemma 5.4), part (ii) of Lemma 3.1 and the established part (i) show that the vertices of $P^{L,U} = H^{\rho_*^{L,U}} = C^{\rho_*^{L,U}}$ are the $(\rho_*^{L,U})_\sigma$ with σ ranging over $\sum^p$, and part (i) of Lemma 5.2 shows that these are precisely the $\rho(\pi_\sigma)'$s.

(iii) Part (iii) of Lemma 3.1, the supermodularity of $\rho_*^{L,U}$ (Lemma 5.4) and (the established) part (i), immediately imply that the directions of the edges of $P^{L,U} = H^{\rho_*^{L,U}} = C^{\rho_*^{L,U}}$ are proportional to differences of unit vectors. □

The equality $P^{L,U} = C^{\rho_*^{L,U}}$ in part (i) of Theorem 5.5 provides a linear inequality representation for $P^{L,U}$ in terms of the $\rho_*^{L,U}(I)$s, using p variables and 2^p constraints. Now, the method for determining the $\rho_*(I)'$s in Section 4 (following Theorem 4.3) can be modified for computing the $\rho_*^{L,U}(I)$s, with (5.5) replacing (4.3). As (5.5) uses the $n_u^-(I)'$s instead of using the n_u's in (4.3), the only needed modification is the computation of the $n_u^-(I)'$s (instead of the $n_u(I)'$s). But, these are available from (5.3) – their computation requires the computation of all $(n_u - \sum_{i\in I} L_{ui})'$s and $(\sum_{i\in I} U_{ui})'$s and executing a minimization of two terms for each I. So, the $n_u^-(I)'$s can be determined with at most $3t2^p$ arithmetic operations instead of the $t2^p$ operations needed to compute the $n_u(I)'$s. The resulting effort to compute the $\rho_*^{L,U}(I)$s is then bounded by $\sum_{u=1}^t n_u + 4t2^p$ arithmetic operations. With p fixed, the bound is linear in the number of components $\sum_{u=1}^t n_u$.

Part (i) of Theorem 5.5 and part (i) of Lemma 5.2 show that each vertex of $P^{L,U}$ has the representation $\rho(\pi_\sigma)$ for some permutation σ of $\{1,\dots,p\}$; in particular, the number of vertices of $P^{L,U}$ is bounded by the number of permutations over $\{1,\dots,p\}$, namely by $p!$. Once all the $n_u^-(I)'$s and $(\rho_*^{L,U})(I)'$s are determined, the π_σ's are readily available and the computation of each $\rho(\pi_\sigma) = (\rho_*^{L,U})_\sigma$ from (3.1) with $\lambda = \rho_*^{L,U}$, requires t subtractions. This procedure will generate a set of p! (monotone) assemblies along with the associated vectors such that these vectors are precisely the vertices of $P^{L,U}$; the additional computational effort, beyond the computation of the $n_u^-(I)'$s and the $(\rho_*^{L,U})(I)'$s, consists of $p!t$ arithmetic operations.

As the π_σ's are monotone assemblies, part (ii) of Theorem 5.5 implies that each vertex of $P^{L,U}$ has a representation as the vector that is associated with some monotone assembly. But, the next example demonstrates that the equality of the set of vertices of the assembly polytope and the set of vectors associated with monotone assemblies does not extend from the prescribed-sizes problem (part (ii) of Theorem 4.3) to the bounded-sizes problem.

Example 1 Suppose $t=1$, $p=2$, $n_1=4$, $\rho_{1i}=-1$ for $i=1,\dots,4$. Consider the set of assemblies π with $1\leqslant|\pi_{11}|\leqslant 3$ and $1\leqslant|\pi_{12}|\leqslant 3$. The assemblies in this set are the (ordered) partitions of 1, 2, 3, 4 into two nonempty parts. Consider the monotone assemblies $\pi^0 = (\{1,2\},\{3,4\}), \pi^1 = (\{1,2,3\},\{4\},)$ and $\pi^2 = (\{1\},\{2,3,4\})$. We then have that $\rho(\pi^0) = -(2,2), \rho(\pi^1) = -(3,1)$ and $\rho(\pi^2) = -(1,3)$ and $\rho(\pi^0) = \frac{1}{2}\rho(\pi^1) + \frac{1}{2}\rho(\pi^2)$, assuring that $\rho(\pi^0)$ is not a vertex of the corresponding assembly polytope. □

We next use Theorem 5.5 to solve the optimal assembly problem.

COROLLARY 5.6 *There exists a permutation σ such that π_σ is a monotone optimal assembly.*

Proof. Proposition 2.1, the asymmetric Schur convexity of g and part (iv) of Theorem 5.5 assure that g attains a maximum over $P^{L,U}$ at a vertex of that polytope. Part (ii) of Theorem 5.5 assures that such a vertex has a representation $\rho(\pi_\sigma)$. Now, let π be an arbitrary assembly; then $\rho(\pi) \in P^{L,U}$ and $R(\pi) = g[\rho(\pi)] \leqslant g[\rho(\pi_\sigma)] = R(\pi_\sigma)$. Part (ii) of Theorem 5.5 also assures that π is monotone. □

The two paragraphs following Theorem 5.5 showed how the set of vertices of $P^{L,U}$ can be generated with computational effort that is linear in n (the size of the partitioned set), while proportional to $p!$. By Theorem 5.6, the evaluation of $R(.)$ for these assemblies and the selection of the best one yields an optimal solution to the optimal assembly problem with bounded part-sizes.

6. Discussion and Extensions

The equality $P^{L,U} = C^{\rho_*^{L,U}}$, established in Theorem 5.5, provides a representation of $P^{L,U}$ as the feasible set of a system of 2^P linear inequalities with p variables; it follows that when p is small, linear functions can be efficiently optimized over $P^{L,U}$. This conclusion together with part (iii) of Theorem 5.5 allow one to apply the vertex enumeration method that was developed in [17] for solving (the newly defined) Convex Combinatorial Optimization Problems. Specifically, [17] describes an algorithm that will efficiently enumerate vertices of a polytope P under two assumptions: (i) the efficient solvability of linear programs over P, and (ii) the availability of a (short) list of vectors that contains directions of all of P's edges. Theorem 5.5 establishes these properties for the assembly polytope. We have already seen that linear functions can be efficiently optimized over $P^{L,U}$ and part (iii) of Theorem 5.5 identifies $\binom{p}{2}$ vectors which cover the directions of all the edges of $P^{L,U}$. The algorithm that is described in [17] can then enumerate the vertices of $P^{L,U}$ by solving at most $O[p^{\binom{p}{2}}]$ linear programs over $P^{L,U}$. Unfortunately, the complexity bound of this method is not better than that of enumerating the π_σs.

Theorem 5.6 and 4.4 provide conditions for the existence of optimal assembly that have some restricted structure. We next discuss conditions under which every optimal assembly has that structure. We say that a performance function is *coherent* if no module i is irrelevant in the sense that $J(.)$ is independent of the ith variable. Following [12, example 3] we observe that when $J(.)$ is coherent the function $g(.)$ defined by (2.3) is strictly asymmetric Schur convex on $(-\infty, 0)^p$.[1] It then follows from

[1]It was observed in [11] that coherence suffices for g to be strictly assymmetric Schur convex, but the statement therein ignores the fact that the conclusion applies only to the restriction of g to $(-\infty, 0)^p$.

Proposition 2.1 that every maximizer of $g(.)$ over $P^{L,U}$ is a vertex of $P^{L,U}$, implying that every optimal assembly π has $\rho(\pi)$ as a vertex if $P^{L,U}$. We further observe that the (rather complicated) analysis of [1] can be used to show that when all the inequalities of (2.1) hold strictly, each vertex of $P^{L,U}$ corresponds to a unique assembly. It then follows from the results of Section 5 that every optimal assembly is monotone, has the form π_σ and has its associated vector as a vertex of $P^{L,U}$.

We finally describe extensions of our results to assembly problems in which the objective function does not necessarily express system-reliability. Consider an assembly problem in which ρ_{ui}'s are associated with the partitioned elements, but without having the interpretation of element-reliability. Also, the objective function $R(.)$ is expressible as $g[\rho(\pi)]$ with $\rho(\pi)_i$'s defined by the right-hand side of (2.2), without $R(.)$ having an interpretation of system-reliability. We observe that the results and methods of Section 5 (and 4) extend to such assembly problems when $g(.)$ is any asymmetric Schur convex function on $P^{L,U}$ and the ρ_{ui}'s are arbitrary numbers that satisfy (2.1). In particular, the nonpositivity of the ρ_{ui}'s is needed for the representation of $\rho_*^{L,U}$ through (5.3) (and its instance (4.3) that applies to the prescribed-sizes case). Still, these formulae – (5.3) and (4.3) – do have simple counterparts when nonnegativity replaces the nonpositivity, that is, (2.1) is replaced by

$$-\infty < \rho_{u1} < \rho_{u2} < \cdots < \rho_{un_u} < 0 \text{ for } u = 1, \ldots, t. \tag{6.1}$$

Consequently, one can derive counterparts of the results of Section 5, in particular, $P^{L,U} = C^{\rho_*^{L,U}}$, (providing a representation of $P^{L,U}$ through a system of linear inequalities), the vertices of $P^{L,U}$ are the vectors associated with simple monotone assemblies and directions of edges of $P^{L,U}$ are proportional to differences of standard unit vectors. When g is strictly asymmetric Schur convex, and the inequalities in (2.1) or (6.1) are strict, we get that every optimal assembly has the special structure.

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